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Coisotropic branes, noncommutativity, and the mirror correspondence

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Abstract: We study coisotropic A-branes in the sigma model on a four-torus by explicitly constructing examples. We find that morphisms between coisotropic branes can be equated with a fundamental representation of the noncommutatively deformed algebra of functions on the intersection. The noncommutativity parameter is expressed in terms of the bundles on the branes. We conjecture these findings hold in general. To check mirror symmetry, we verify that the dimensions of morphism spaces are equal to the corresponding dimensions of morphisms between mirror objects.

Keywords: D-branes, Non-Commutative Geometry.

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1. Introduction and summary

Kapustin and Orlov [6] discovered coisotropic A-branes in supersymmetric sigma models and speculated about the role they played in the Fukaya category.[1] In this paper, we study these branes in examples, by explicitly constructing the boundary superconformal field theories they define. Our examples reveal that the morphism spaces between coisotropic branes are representations of a noncommutative algebra. This algebra is found to be the quantization of the algebra of functions on the intersection of the coisotropic branes. The representation is the fundamental representation of the algebra, and generalizes the definition of lagrangian intersections (where the algebra is trivial at each intersection point). We conjecture this to be true in general.

More precisely, we study coisotropic branes on a four-torus A appearing at special points in Kähler moduli space. Writing E^τ for $\mathbb{R}^2/\mathbb{Z}^2$ with (complexified) symplectic form $\omega = \tau dx \wedge dy$ and E_τ for its mirror $\mathbb{C}/\Lambda_\tau$, $\Lambda_\tau = \{n + m\tau\}$, we take $A = E^\tau \times E^\tau$ with symplectic form $\pi_1^*\omega + \pi_2^*\omega$, where π_i are the two projections. The mirror $\widetilde{A}$ is given by $\widetilde{A} = E_\tau \times E_\tau$. When τ is an imaginary quadratic extension E_τ has complex

[1] It has been notoriously difficult to define the Fukaya category appropriately and to describe its general objects – see [4] [10].

multiplication by $\alpha = n\tau$, where n is determined by τ.[2] At such a point $\widetilde{A}$ attains a divisor $D = (z_2 = \alpha z_1)$ indicating a jump in the dimension of the Picard group, i.e. a new line bundle $[D]$ appears ($[D]$ does not extend to a bundle over a nontrivial complex family containing $\widetilde{A}$). Correspondingly, A should acquire a new A-brane, (C, V), where C is coisotropic and V is a U(1) line bundle on C.

Now as $[D]$ is not (immediately) expressible in terms of pull-backs of line bundles by the projections, its mirror C is thought to be a coisotropic brane, and in [6] a Chern class calculation was made to verify this point of view. In that paper the question of of defining Fukaya-type morphisms for coisotropic branes was also raised.

Here we explicitly construct the boundary field theory, solve the boundary conditions, quantize and find the ground states. These ground states correspond to morphisms in the Fukaya-type category of branes, and by studying coisotropic-coisotropic or lagrangian-coisotropic boundary conditions at the two endpoints we can gain a concrete understanding of the mysterious coisotropic branes in this example. For example, $\mathrm{Hom}((C, -V^*), (C, V))$ is found to be the fundamental representation of the noncommutative algebra of functions on A_q, with noncommutativity parameter q determined by the curvature of V. (Although the appearance of noncommutativity in the presence of curvature is not entirely unexpected, the novelty here is that the boundary conditions can be different on the two ends of the string, as opposed to a B-field.) Finite-dimensionality of this representation is related to integrality of the curvature form.

2. Coisotropic boundary conditions

In this section we will construct a supersymmetric sigma model with target space a flat, symplectic four-torus on a Riemann surface that is an infinite strip with two boundary components. The boundary conditions correspond to dual coisotropic A-branes at the two ends. We find mode expansions solving the equations of motion and quantize to find the corresponding mode operators. The explicit form of the hamiltonian and supersymmetry generators Q makes it easy to read off the states of Q-cohomology.

2.1 Example: $\tau = i$

Consider the SUSY sigma model with world-sheet $\Sigma = [0, \pi] \times \mathbb{R}$ and target the flat square torus of real dimension four with coordinates $X^\mu \sim X^\mu + 2\pi$. The action is

$$
\begin{aligned}
S = \frac{1}{4\pi} \int_\Sigma dtds \left\{ (\partial_t X^\mu)^2 - (\partial_s X^\mu)^2 + i\theta^\mu_+ \partial_- \theta^\mu_+ + i\theta^\mu_- \partial_+ \theta^\mu_- \right\} + \\
+ \frac{1}{2\pi} \int_{s=\pi} dt A_\mu \partial_t X^\mu + \frac{1}{2\pi} \int_{s=0} dt A_\mu \partial_t X^\mu \,,
\end{aligned}
$$

where $\theta^\mu_\pm$ are real spinors and

$$
A_1 = \begin{cases} X^3 & 0 \le X^3 \le \pi \\ X^3 - 2\pi & \pi < X^3 < 2\pi \end{cases} \qquad A_2 = \begin{cases} -X^4 & 0 \le X^4 \le \pi \\ -X^4 + 2\pi & \pi < X^4 < 2\pi \end{cases} \quad A_3 = A_4 = 0
$$

[2]For complex multiplication by $\widetilde{\alpha}$ we require $\widetilde{\alpha} \cdot \Lambda \subset \Lambda$, so in particular $\widetilde{\alpha} \cdot 1 = m + n\tau$, which is true if and only if we have complex multiplication by $\alpha = n\tau$. The second lattice generator gives $(n\tau)\tau = -p + q\tau$, whence $\tau = \frac{1}{2n}(q + i\sqrt{4pn - q^2})$, where $4pn > q^2$ (otherwise $\Lambda \subset \mathbb{R} \subset \mathbb{C}$).

are abelian gauge fields. The boundary term is equivalent to two target-filling D-branes with opposite U(1) connection

$$A = \pm \frac{i}{2\pi}\left[A_1 dX^1 + A_2 dX^2\right]$$

and curvature

$$F = dA = \pm \frac{1}{2\pi}\left(-dX^1 \wedge dX^3 + dX^2 \wedge dX^4\right).$$

With respect to the symplectic form

$$\omega = \frac{1}{2\pi}\left[dX^1 \wedge dX^2 + dX^3 \wedge dX^4\right]$$

the A-brane condition of [6] is satisfied

$$F\omega^{-1}F + \omega = 0\,.$$

2.1.1 Equations of motion and boundary conditions

With respect to a variation $X^\mu \to X^\mu + \delta X^\mu$, the the boundary variation of the action is

$$\begin{aligned}\delta S^B_{\rm bdry} = & \frac{1}{2\pi}\int_{s=0} dt\left\{\delta X^1(-\partial_t X^3 - \partial_s X^1) + \delta X^2(\partial_t X^4 - \partial_s X^2) + \right.\\ & \left. + \delta X^3(\partial_t X^1 - \partial_s X^3) + \delta X^4(-\partial_t X^2 - \partial_s X^4)\right\} + \\ & + \frac{1}{2\pi}\int_{s=\pi} dt\left\{\delta X^1(-\partial_t X^3 + \partial_s X^1) + \delta X^2(\partial_t X^4 + \partial_s X^2) + \right.\\ & \left. + \delta X^3(\partial_t X^1 + \partial_s X^3) + \delta X^4(-\partial_t X^2 + \partial_s X^4)\right\}.\end{aligned}$$

Therefore, bosons must satisfy the wave equation $(\partial_s^2 - \partial_t^2)X^\mu = 0$ with boundary conditions

$$\begin{array}{lllll} s=0: & \partial_s X^1 = \partial_t X^3\,; & \partial_s X^2 = -\partial_t X^4\,; & \partial_s X^3 = -\partial_t X^1\,; & \partial_s X^4 = \partial_t X^2 \\ s=\pi: & \partial_s X^1 = -\partial_t X^3\,; & \partial_s X^2 = \partial_t X^4\,; & \partial_s X^3 = \partial_t X^1\,; & \partial_s X^4 = -\partial_t X^2\,. \end{array}$$

A fermionic variation $\theta^\mu \to \theta^\mu + \delta\theta^\mu$ induces a boundary variation

$$\delta S^F_{\rm bdry} = \frac{i}{4\pi}\int_{s=0} dt\left\{-\theta^\mu_+\delta\theta^\mu_+ + \theta^\mu_-\delta\theta^\mu_-\right\} + \frac{i}{4\pi}\int_{s=\pi} dt\left\{\theta^\mu_+\delta\theta^\mu_+ - \theta^\mu_-\delta\theta^\mu_-\right\}.$$

The equation of motion for the fermions is the Dirac equation $\partial_\pm \theta^\mu_\mp = 0$ and to impose the boundary conditions we require [3] invariance under $N=1$ SUSY with generators

$$\delta X^\mu = i\epsilon(\theta^\mu_+ + \theta^\mu_-)\,; \qquad \delta\theta^\mu_\pm = -\epsilon\partial_\pm X^\mu$$

and deduce

$$\begin{array}{lllll} s=0: & \theta^1_+ = \theta^3_-\,; & \theta^2_+ = -\theta^4_-\,; & \theta^3_+ = -\theta^1_-\,; & \theta^4_+ = \theta^2_- \\ s=\pi: & \theta^1_+ = -\theta^3_-\,; & \theta^2_+ = \theta^4_-\,; & \theta^3_+ = \theta^1_-\,; & \theta^4_+ = -\theta^2_-\,. \end{array}$$

Moreover, bosonic and fermionic boundary conditions are invariant under N=2 supersymmetry with generators

$$\delta X^\mu = \epsilon\theta^\mu_- - \bar\epsilon\theta^\mu_+\,; \qquad \delta\theta^\mu_+ = i\epsilon\partial_+ X^\mu\,; \qquad \delta\theta^\mu_- = -i\bar\epsilon\partial_- X^\mu$$

and supercharge

$$Q = \frac{1}{2\pi}\int_0^\pi ds(\theta_+^1 - i\theta_+^2)(\partial_+X^1 + i\partial_+X^2) + (\theta_-^1 + i\theta_-^2)(\partial_-X^1 - i\partial_-X^2) + \\ + (\theta_+^3 - i\theta_+^4)(\partial_+X^3 + i\partial_+X^4) + (\theta_-^3 + i\theta_-^4)(\partial_+X^3 - i\partial_-X^4)\,.$$

2.1.2 Mode expansions

Under the change of variables

$$\begin{aligned} Y^1(t,s) &= \frac{1}{\sqrt{2}}(X^1(t,s) + iX^3(t,s))\,; \qquad Y^2(t,s) = \frac{1}{\sqrt{2}}(X^2(t,s) + iX^4(t,s)) \\ \eta_\pm^1(t,s) &= \frac{1}{\sqrt{2}}(\theta_\pm^1(t,s) + i\theta_\pm^3(t,s))\,; \qquad \eta_\pm^2(t,s) = \frac{1}{\sqrt{2}}(\theta_\pm^2(t,s) + i\theta_\pm^4(t,s)) \end{aligned}$$

the boundary conditions decouple as

$$\begin{array}{ll} s=0: & \partial_sY^1 = i\partial_sY^1\,; \quad \partial_sY^2 = -i\partial_sY^2\,; \quad \eta_-^1 = i\eta_+^1\,; \quad \eta_-^2 = -i\eta_+^2 \\ s=\pi: & \partial_sY^1 = -i\partial_tY^1\,; \quad \partial_sY^2 = i\partial_tY^2\,; \quad \eta_-^1 = -i\eta_+^1\,; \quad \eta_-^2 = i\eta_+^2\,. \end{array}$$

The corresponding conjugate momenta are

$$P^k := \frac{\partial\mathcal{L}}{\partial\partial_tY^k} = \frac{1}{2\pi}\partial_t\overline{Y}^k + \frac{i}{4\pi}(Y^k - \overline{Y}^k + (\cdots))(\alpha_k\delta(s) + \beta_k\delta(\pi - s))\,, \tag{2.1}$$

where $\alpha_1 = \beta_1 = -\alpha_2 = -\beta_2 = -1$ and $(\cdots)$ stands for the possible 2π factor in the piecewise definition of the gauge fields. We adapt to our case the mode expansion of [1]

$$Y^k(t,s) = y_k + i\left[\sum_{n=1}^\infty a_{k,n}\,\zeta_n^k(t,s) - \sum_{m=0}^\infty b_{k,m}^\dagger\,\zeta_{-m}^k(t,s)\right],$$

where

$$\zeta_n^k(t,s) := |n - \epsilon_k|^{-1/2}\cos\left[(n-\epsilon_k)s + \tan^{-1}\alpha_k\right]e^{-i(n-\epsilon_k)t}$$

and $\pi\epsilon_k = \tan^{-1}\alpha_k + \tan^{-1}\beta_k$. Notice that boundary conditions are satisfied and the functions ζ_n^k form a complete orthogonal system with respect to the inner product

$$\int_0^\pi \frac{ds}{\pi}\,\overline{\zeta}_n^k\left[i\overleftrightarrow{\partial}_t + \alpha_k\delta(s) + \beta_k\delta(\pi - s)\right]\zeta_m^k = \delta_{mn}\mathrm{sign}\,(n - \epsilon_k)\,.$$

Moreover, each ζ_n^k is orthogonal to the zero mode in the sense that

$$\int_0^\pi \frac{ds}{\pi}[i\partial_t + \alpha_k\delta(s) + \beta_k\delta(\pi - s)]\zeta_n^k = 0\,.$$

Using these relations, one can invert the mode expansions

$$\begin{aligned} a_{k,n} &= \int_0^\pi \frac{ds}{\pi}\overline{\zeta_n^k}\left[\overleftrightarrow{\partial}_t - i\alpha_k\delta(s) - i\beta_k\delta(\pi - s)\right]Y^k \\ b_{k,n}^\dagger &= \int_0^\pi \frac{ds}{\pi}\overline{\zeta_{-n}^k}\left[\overleftrightarrow{\partial}_t - i\alpha_k\delta(s) - i\beta_k\delta(\pi - s)\right]Y^k \\ y_k &= \frac{1}{\alpha_k + \beta_k}\int_0^\pi\left[i\partial_t + \alpha_k\delta(s) + \beta_k\delta(\pi - s)\right]Y^k\,. \end{aligned} \tag{2.2}$$

For the mode expansion of the fermions, we follow [3, sec. 39.1.2.5]

$$\eta_\pm^k = \sum_{n\in\mathbb{Z}}\eta_{k,n}e^{\pm i[(n-\epsilon_k)(s\pm t) + \tan^{-1}\alpha_k]}\,.$$

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2.1.3 Quantization

We impose the equal-time canonical commutation relations

$$\left[Y^k(t,s), P^j(t,s')\right] = \left[\overline{Y}^k(t,s), \overline{P}^j(t,s')\right] = \left\{\eta^k_\pm(t,s), \eta^j_\pm(t,s')\right\} = i\delta_{kj}\delta(s-s')$$

from which, using (2.1) and (2.2), we deduce

$$\begin{aligned}\left[y_k, y_k^\dagger\right] &= \frac{\pi i}{(\alpha_k+\beta_k)^2}\int_0^\pi\int_0^\pi ds ds'(\alpha_k\delta(s')+\beta_k\delta(\pi-s'))\left[\overline{P}^k(t,s),\overline{Y}^k(t,s')\right] + \\ &\quad + (\alpha_k\delta(s)+\beta_k\delta(\pi-s))\left[Y^k(t,s), -P^k(t,s')\right] \\ &= \frac{2\pi}{\alpha_k+\beta_k}\,.\end{aligned}$$

Similarly,

$$\left[b_{k,n}, b^\dagger_{k,m}\right] = \delta_{nm} = \left[a_{k,n}, a^\dagger_{k,m}\right].$$

The nontrivial fermionic anticommutators are (see e.g. [3, sec. 39.1.2.25])

$$\{\eta_{k,n}, \eta^\dagger_{k,m}\} = \delta_{nm}\,.$$

The mode expansion for the hamiltonian is

$$\begin{aligned}H &= \frac{1}{\pi}\int_0^\pi ds\,|\partial_t Y^k|^2 + |\partial_s Y^k|^2 + \frac{i}{2}\left(\overline{\eta}^k_+ \overleftrightarrow{\partial}_s \eta^k_+ - \overline{\eta}^k_- \overleftrightarrow{\partial}_s \eta^k_-\right) \\ &= \sum_{n=1}^\infty\left(n+\frac{1}{2}\right)a^\dagger_{1,n}a_{1,n} + \sum_{m=0}^\infty\left(m-\frac{1}{2}\right)b^\dagger_{1,m}b_{1,m} + \sum_{n=0}^\infty\left(n-\frac{1}{2}\right)a^\dagger_{2,n}a_{2,n} + \\ &\quad + \sum_{m=1}^\infty\left(m+\frac{1}{2}\right)b^\dagger_{2,m}b_{2,m} + \sum_{n=1}^\infty\left(n+\frac{1}{2}\right)\eta^\dagger_{1,n}\eta_{1,n} + \sum_{m=0}^\infty\left(m-\frac{1}{2}\right)\eta_{1,-m}\eta^\dagger_{1,-m} + \\ &\quad + \sum_{n=0}^\infty\left(n-\frac{1}{2}\right)\eta^\dagger_{2,n}\eta_{2,n} + \sum_{m=1}^\infty\left(m+\frac{1}{2}\right)\eta_{2,-m}\eta^\dagger_{2,-m}\,.\end{aligned}$$

Notice that the zero-modes commute with the hamiltonian. Using $\sqrt{2}y_1 = x_1 + ix_3$, the commutation relation for the bosonic zero-modes can be rewritten as

$$[x_1, x_3] = -i\pi\,. \tag{2.3}$$

Therefore, we can think of

$$p_1 := -\frac{x_3}{\pi}$$

as the conjugate momentum for x_1 and use it to label ground states. As in [1], compactness of the target implies the existence of finitely many ground states. In fact, on the one hand, the zero-mode wave function

$$\langle x_1|p_1\rangle = e^{ip_1x_1}$$

has to remain single-valued under $x_1 \to x_1 + 2\pi$ and this implies $p_1 \in \mathbb{Z}$. On the other hand, $x_3 \to x_3 + 2\pi$ implies $p_1 \to p_1 - 2$ so that there are only two independent zero-mode wave functions. A similar analysis can be repeated for the other zero-mode y_2 so that overall

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there are four ground states of the form $|p_1, p_2\rangle$ corresponding to the values $p_1 = 0, 1$ and $p_2 = 0, 1$. Using the relation $2H = \{Q, Q^\dagger\}$, we see that excited states are killed by the BRST operator. In particular fermionic modes do not contribute to the cohomology as, due to the fractional mode expansion, they have either positive or negative energy. Therefore, the dimension of the BRST cohomology ring is 4.

Another point of view on the space of ground states is as follows. Define $U = e^{ix_1}$ and $V = e^{i\pi p_1} = e^{ix_3}$. Then the relation (2.3) together with periodicity defines the algebra

$$U^2 = V^2 = 1\,, \qquad UV = -VU\,.$$

This algebra is the noncommutative two-torus! The fundamental representation is two dimensional and can be constructed from the x_1-momentum eigenvectors $e_1 = |p_1 = 0\rangle, e_2 = |p_1 = 1\rangle$, with U and V acting as matrices

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that U increases the p_1 eigenvalue by 1 (mod 2) and we have made V diagonal by the basis choice. The states x_2 and x_4 define another copy of this algebra. The ground states $|i, j\rangle$ are identified with $e_i \otimes e_j$. The dimension of the representation of the ground state algebra, 4, agrees with the mirror calculation performed in section 3.

2.2 Coisotropic-coisotropic and lagrangian-coisotropic examples

We consider some examples along the general lines of the previous section, abbreviating the the analysis somewhat.

2.2.1 Coisotropic-coisotropic boundary conditions

If we fix the symplectic structure to be the standard one ω, then a skew-symmetric 4×4 matrix $F = (a_{ij})$ with integer entries is the curvature of a coisotropic A-brane if and only if

$$F \wedge \omega = 0\,; \qquad \mathrm{Pfaff}(F) = \frac{1}{2}\int F \wedge F = 1\,.$$

Notice that if F is of this form then so is $-F$, and one can study the open string states from $-F$ to F as before. Therefore, our computation works for any such pair $(-F, F)$, the results depending only on the determinant of the curvature. When the curvatures F, G of the branes at the endpoints are not multiples of each other, we need to modify the mode expansion. Consider for example the case

$$\begin{aligned} s = 0: &\quad F = -J = dX^1 \wedge dX^3 - dX^2 \wedge dX^4 \\ s = \pi: &\quad G = K = -dX^1 \wedge dX^4 - dX^2 \wedge dX^3\,. \end{aligned}$$

This notation is suggested by the familiar quaternionic relations

$$I^2 = J^2 = K^2 = -1\,; \qquad IJ = K\,,$$

where $I = \omega^{-1} g$ (here g is the identity matrix). It is convenient to introduce the quaternionic field

$$Q(s,t) = X^1(s,t) + IX^2(s,t) + JX^3(s,t) + KX^4(s,t)$$

with equation of motion $(\partial_s^2 - \partial_t^2)Q = 0$ and boundary conditions

$$\partial_s Q(0,t) = -J\partial_t Q(0,t)\,; \qquad \partial_s Q(\pi,t) = K\partial_t Q(\pi,t)\,.$$

We can solve the boundary conditions with the mode expansion

$$Q(s,t) = q_0 + \sum_{n\in\mathbb{Z}} e^{-I(n+\frac{1}{4})s} e^{K(n+\frac{1}{4})t} q_n\,.$$

In analogy with the previous example,

$$q_0 = x_1 + Ix_2 + Jx_3 + Kx_4 = \frac{J+K}{2}\int_0^\pi ds\,[\partial_t - J\delta(s) - K\delta(\pi - s)]\,Q(s,t)\,.$$

Using

$$\begin{aligned} P^1 &= \frac{1}{2\pi}\partial_t X^1 + \frac{1}{2\pi}X^3\delta(s) + \frac{1}{2\pi}X^4\delta(\pi-s)\,; \qquad P^3 = \frac{1}{2\pi}\partial_t X^3 \\ P^2 &= \frac{1}{2\pi}\partial_t X^2 - \frac{1}{2\pi}X^4\delta(s) + \frac{1}{2\pi}X^3\delta(\pi-s)\,; \qquad P^4 = \frac{1}{2\pi}\partial_t X^4 \end{aligned}$$

we deduce

$$\begin{aligned} x_1 &= \frac{1}{2}\int_0^\pi \left(-2\pi P^3 - 2\pi P^4 + (X^1 - X^2)\delta(s) + (X^1 + X^2)\delta(\pi - s)\right) \\ x_2 &= \frac{1}{2}\int_0^\pi \left(2\pi P^3 - 2\pi P^4 + (X^1 + X^2)\delta(s) + (-X^1 + X^2)\delta(\pi - s)\right) \\ x_3 &= \frac{1}{2}\int_0^\pi \left(2\pi P^1 + 2\pi P^2\right) \\ x_4 &= \frac{1}{2}\int_0^\pi \left(2\pi P^1 - 2\pi P^2\right). \end{aligned}$$

The canonical commutation relations imply the non-vanishing commutators

$$\begin{aligned} [x_1, x_3] &= \pi i\,; \qquad [x_1, x_4] = \pi i \\ [x_2, x_3] &= \pi i\,; \qquad [x_2, x_4] = -\pi i\,. \end{aligned}$$

Note that $[x_j, x_k] = ((G-F)^{-1})_{jk}$. As we will see in section 2.3, these relations imply that there are two ground states.

2.2.2 Lagrangian-coisotropic boundary conditions

The case where only one of the branes is strictly coisotropic can be treated in a similar way. As a simple example, suppose that the bosonic part of the action is

$$S = \frac{1}{4\pi}\int_\Sigma dtds\{(\partial_t X^\mu)^2 - (\partial_s X^\mu)^2\} + \frac{1}{2\pi}\int_{s=\pi} A_\mu \partial_t X^\mu dt$$

and we impose boundary conditions

$$
\begin{aligned}
&s = 0 : X^2 = 0\,; \quad \partial_s X^1 = 0\,; \quad X^4 = 0\,; \quad \partial_s X^3 = 0 \\
&s = \pi : \partial_s X^1 = -\partial_t X^3\,; \quad \partial_s X^2 = \partial_t X^3\,; \quad \partial_s X^3 = \partial_t X^1\,; \quad \partial_s X^4 = -\partial_t X^2\,.
\end{aligned}
$$

Since X^2 and X^4 have no interesting zero modes, we focus on complex solutions

$$
Y(s,t) = \frac{1}{\sqrt{2}}(X^1(s,t) + iX^3(s,t)) \quad \text{s.t.} \quad \partial_s Y(0,t) = 0\,; \quad \partial_t Y(\pi,t) = i\partial_s Y(\pi,t)\,.
$$

We use the expansion

$$
Y(s,t) = y + \sum_{n\in\mathbb{Z}} a_n \cos\left[\left(n - \frac{1}{4}\right)s\right] e^{-i(n-1/4)t}
$$

and in analogy with the above computation

$$
\left[y, y^\dagger\right] = \left[\int_0^\pi ds\{i\partial_t + \delta(s-\pi)\}Y(s,t), \int_0^\pi ds\{-i\partial_t + \delta(s-\pi)\}\overline{Y}(s,t)\right] = -2\pi\,.
$$

Therefore, $[x_1, x_3] = -2\pi i$ and we have a single bosonic state.

2.3 Noncommutativity in the general case

For the general torus $\mathbb{R}^{2d}/(2\pi\mathbb{Z})^{2d}$, we consider coisotropic branes with bundles $\mathcal{E}_1$ at $s = 0$ and $\mathcal{E}_2$ at $s = \pi$, with curvatures defined by the integer, skew-symmetric matrices F_{ij} and G_{ij}, respectively. We will now argue that the zero modes obey the following commutation relations:

$$
[x^k, x^l] = 2\pi i A^{kl}\,, \tag{2.4}
$$

where $A = G - F$ is also an integer-valued, skew-symmetric $2d\times 2d$ matrix, which we assume to be invertible,[3] while A^{kl} are the components of the inverse matrix: $A^{kl}A_{lm} = \delta^k{}_m$. To see this, suppose that ζ is a solution to the boundary problem

$$
(\partial_s^2 - \partial_t^2)\zeta(s,t) = 0\,; \qquad \partial_s\zeta(0,t) = F\partial_t\zeta(0,t)\,; \qquad \partial_s\zeta(\pi,t) = G\partial_t\zeta(\pi,t)
$$

and moreover that its t-antiderivative ψ is also a solution to the same problem (this is reasonable since F and G are constant matrices). Under these assumptions,

$$
\int_0^\pi ds[\partial_t + F\delta(s) - G\delta(\pi - s)]\zeta = \partial_s(\psi(\pi) - \psi(s)) + F\partial_t\psi(0) - G\partial_t\psi(\pi) = 0 \tag{2.5}
$$

i.e. we have an inner-product which makes all possible oscillators orthogonal to the zero-modes. Therefore, the components of the zero-mode of the field X are

$$
x^l = \frac{a^{ml}}{2}\int_0^\pi 4\pi P^m + (f_{mn}\delta(s) - g_{mn}\delta(\pi - s))X^n\,.
$$

[3] This assumption is violated, for example, when one considers open strings between a coisotropic brane and itself. In such an example, there are fermionic zero modes and the ground state sector is not simply given by the zero-mode algebra. A unified treatment of coisotropic branes should yield a mathematical cohomology theory matching BRST cohomology, as Gualtieri's d_L operator does for the endomorphisms of coisotropic branes *qua* generalized complex submanifolds [5].

We conclude

$$\left[x^k, x^l\right] = a^{jk} a^{ml} \left[2\pi P^j + \frac{1}{2} a_{mj} X^m, 2\pi P^m + \frac{1}{2} a_{jm} X^j\right] = 2\pi i a^{kl} \,. \tag{2.6}$$

(Similarly, one can compute the zero-mode algebra when the brane with charge F is replaced with the lagrangian brane of 2.2.2. It is enough to discard even-numbered coordinates and set $F = 0$ in (2.5) and (2.6). Moreover, a rotation of the coordinates sends a coisotropic brane to another coisotropic one so that the same computation works for a linear lagrangian placed at $X^{2k}(0) = m_k X^{2k-1}(0)$ where $m_k \in \mathbb{Q}$, $k = 1, \ldots, d$). We assume that our coordinates are A-symplectic in the sense that $A_{ab} = 0$, $a, b = 1 \ldots d$ and similarly for $A_{d+a,d+b}$. In fact, define $u^a = x^a$, $v^a = x^{d+a}$. Therefore, A is defined by the $d \times d$ integer matrix N, i.e. A has block-anti-diagonal form

$$A = \begin{pmatrix} 0 & N \\ -N^T & 0 \end{pmatrix} .$$

Note

$$A^{-1} = \begin{pmatrix} 0 & -N^{-T} \\ N^{-1} & 0 \end{pmatrix} .$$

Then from the relations (2.4), we find the corresponding momenta and relations

$$p_j = \frac{1}{2\pi} x^k A_{kj} \qquad [p_j, p_k] = -\frac{i}{2\pi} A_{jk}$$

(or $p_{u^a} = -\frac{1}{2\pi} v^b N_{ab}$ and $p_{v^a} = \frac{1}{2\pi} u^b N_{ba}$). Note that the Dirac quantization condition applied to the two-dimensional sub-torus defined by the coordinate pair (jk) says $(2\pi)^2(-1/2\pi)A_{jk} = 2\pi n$, where $n \in \mathbb{Z}$, i.e. A is an integer-valued matrix. So our relations are consistent. Since we can rewrite the momenta as coordinates, the lattice translation in the k-direction, $\vec{x} \to \vec{x} + 2\pi e_k$ implies the corresponding periodicity in the momentum lattice

$$\vec{p} = p_i e^i \sim (p_i + A_{ki}) e^i \,, \qquad k = 1, \ldots, 2d \,,$$

or

$$\vec{p}_u = p_{u^a} e^a \sim (p_{u^a} + N_{ab}) e^b \,, \qquad \vec{p}_v = p_{v^a} e^{d+a} \sim (p_{v^a} + N_{ba}) e^{d+b} \,.$$

Now define the operators

$$U_a = e^{iu^a} \,, \qquad V_a = e^{iv^a} \,,$$

well-defined by the 2π-periodicity of the coordinates. By the form of A, the U operators commute with each other, as do the V operators (so we can diagonalize u-momenta or v-momenta, but not both). Now we interpret the operator U_a as adding a unit of p_{u^a} momentum, whereas V_a adds a unit of p_{v^a} momentum. This imposes the following relations among the U and V

$$\begin{aligned} U_1^{N_{1a}} U_2^{N_{2a}} \ldots U_d^{N_{da}} &= 1 \\ V_1^{N_{a1}} V_2^{N_{a2}} \ldots V_d^{N_{ad}} &= 1 \qquad a = 1, \ldots, d \end{aligned}$$

To write this in a more convenient form, define $U^\alpha = U_1^{\alpha_1} U_2^{\alpha_2} \dots U_d^{\alpha_d}$ for $\alpha \in \mathbb{Z}^d$. Define $N_{a,\bullet} = (N_{a1}, N_{a2}, \dots, N_{ad})$ to be the a-th row of N; similarly define $N_{\bullet,a}$ to be the a-th column. Then the relations of the zero-mode algebra can now be written

$$U^{N_{a,\bullet}} = 1\,, \tag{2.7}$$

$$V^{N_{\bullet,a}} = 1\,, \tag{2.8}$$

$$U_a V_b = e^{2\pi i N^{ba}} V_b U_a\,, \tag{2.9}$$

$$U_a U_b = U_b U_a\,, \tag{2.10}$$

$$V_a V_b = V_b V_a\,, \qquad a, b = 1, \dots, d\,. \tag{2.11}$$

This is a noncommutative torus defined by the matrix N. Note that since N is integer, the components of N^{-1} are rational, so all the noncommutativity parameters are roots of unity. The zero modes naturally define conjugate finite-dimensional u-momentum and v-momentum eigenstate representations, H_u and H_v, by construction. The dimension of the representation corresponds to the index of the sublattice in $\mathbb{Z}^d$ defined by the vectors $N_{a,\bullet}$ (or for the conjugate representation $N_{\bullet,a}$), so

$$\mathrm{dim} H_u = \mathrm{dim} H_v = \mathrm{det} N = \mathrm{Pfaff}(A) = \frac{1}{d!} \int c_1(\mathcal{E}_1^* \otimes \mathcal{E}_2)^d\,.$$

We can now speculate about the general case. Noncommutativity arose from the nonexistence of solutions to the equations of motion that were linear in time. As a result, there were no separate conjugate momenta to the zero-mode oscillators — they formed their own momentum conjugates and did not simultaneously commute. This argument also depended on the decoupling of these modes from the oscillators of the open string. Let us assume such a decoupling in general[4] and study two A-branes (C_1, F_1) and (C_2, F_2) at the left and right boundary conditions. Here C_i can be coisotropic or lagrangian and F_i is the curvature of a connection A_i on a bundle $\mathcal{E}_i \to C_i$. We study the zero modes x^i. The boundary conditions require x^i to lie at the intersection $C_1 \cap C_2$, and we would like to compute the zero-mode algebra. In the case of transverse lagrangian-lagrangian intersection, the zero modes carry no degrees of freedom so their algebra is trivial, except that there is a copy of it at each intersection point. We think of this case thus trivially falling into the noncommutative picture. Now consider the case where at least one brane is coisotropic. We assume that the curvature $F = F_2 - F_1$ on $\mathcal{E}_1^* \otimes \mathcal{E}_2$ is nondegenerate, so there are no fermionic zero-modes and the BRST cohomology is equal to the representation of the zero modes. The natural generalization of (2.4) is that F^{-1} is the Poisson structure of a (Moyal) noncommutativization of the algebra of functions on the intersection $C_1 \cap C_2$. In the examples we computed, this algebra had a "fundamental" representation H, finite-dimensional due to the integrality of the curvature two-form, with

$$\mathrm{dim}_{\mathbb{C}} H = \frac{1}{d!} \int_{C_1 \cap C_2} c_1^d(\mathcal{E})\,,$$

[4]This assumption would follow from an orthogonal basis of solutions to the appropriate Laplace equation (equations of motion) with boundary conditions determined by the curvatures, with respect to an appropriately defined inner product.

where d is half of the real dimension of the intersection. We do not know whether finite dimensionality or the dimension formula holds in the general case. We conjecture this to be the case.

3. B-model calculations

We now wish to describe the mirror B-model calculations of what we have done in previous sections. The cohomology of a holomorphic bundle over a complex torus can be constructed explicitly. We follow the beautiful treatment given in [9].

3.1 Mirror geometry

Let $\widetilde{A} = E_\tau \times E_\tau$ be the product of two elliptic curves with modular parameter $\tau = \tau_1 + i\tau_2$ and let z, w be complex coordinates on the first and second factor respectively. With respect to a standard symplectic basis x_1, y_1, x_2, y_2 for $H^1(\widetilde{A}, \mathbb{Z})$, the complex structure is block diagonal and on each 2×2 factor is given by

$$\mathcal{J} = -\frac{1}{\tau_2}\begin{pmatrix} \tau_1 & |\tau|^2 \\ -1 & -\tau_1 \end{pmatrix} = \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix}^{-1}.$$

The Neron-Severi group $NS(\widetilde{A})$ has three generators L_i such that

$$\begin{aligned} c_1(L_1) &= \frac{1}{\tau_2}\mathrm{Im}((\bar{\tau}x_1 - y_1)(\tau x_1 - y_1)) = x_1 y_1 \\ c_1(L_2) &= \frac{1}{\tau_2}\mathrm{Im}((\bar{\tau}x_2 - y_2)(\tau x_2 - y_2)) = x_2 y_2 \\ c_1(L_3) &= \frac{1}{\tau_2}\mathrm{Im}((\bar{\tau}x_1 - y_1)(\tau x_2 - y_2)) = x_1 y_2 - x_2 y_1 \end{aligned}$$

for generic τ. The first two correspond to pull back divisors under the two projections, while the third appears due to the existence of the diagonal divisor. When E_τ admits complex multiplication by α there is an extra bundle L_4, due to the divisor $w = \alpha z$, with first Chern class multiple of

$$\mathrm{Re}((\bar{\tau}x_1 - y_1)(\tau x_2 - y_2)) = (\tau_1^2 + \tau_2^2)x_1 x_2 + y_1 y_2 - \tau_1(x_1 y_2 + y_1 x_2)\,.$$

It is straightforward to compute the Chern character of the mirror object via Fourier transform. For example, Kapustin and Orlov [6] observed that for $\tau = i$

$$ch(\mathrm{Four}(L_4)) := \int ch(L_4)ch(P) = 1 - l_1 l_2 + y_1 y_2 + l_1 y_1 l_2 y_2\,,$$

where l_1, l_2 are coordinates dual to x_1, x_2, integration is performed with respect to the form $x_1 x_2$ and P is the Poincaré line bundle such that $c_1(P) = x_1 l_1 + x_2 l_2$. In fact, in coordinates

$$l_1 = \frac{1}{2\pi}dX^1\,, \quad l_2 = \frac{1}{2\pi}dX^3\,, \quad y_1 = \frac{1}{2\pi}dX^2\,, \quad y_2 = \frac{1}{2\pi}dX^3\,;$$

this coincides with the Chern character of the coisotropic brane J used in our first example. In a similar way, we interpret the coisotropic brane $-J$ as the Fourier transform of $-[L_4^*]$ i.e. the opposite in $K_0(\widetilde{A})$ of the line bundle dual to L_4. With this identification, one can apply the standard B-model techniques (see e.g. [11, 3]) and use the Grothendieck-Riemann-Roch theorem to compute

$$\dim \mathrm{Ext}^0(-[L_4^*], L_4) = -\chi(L_4^*, L_4) = \int_{\widetilde{A}} ch^2(L_4) = 4\,,$$

in agreement with the A-model calculation. In addition, these elements can be identified with $H^0(L_4^2)$, which naturally carries a representation of the Heisenberg group [9], i.e. the very same noncommutative algebra!

More generally, consider two coisotropic A-branes $\mathcal{E}_1, \mathcal{E}_2$ as in section 2.3. The mirror brane of $\mathcal{E}_1$ has Chern character

$$\begin{aligned} ch(\mathrm{Four}(\mathcal{E}_1)) &= \int ch(\mathcal{E}_1) ch(P) \\ &= -f_{13} + f_{23}x_1y_1 + x_1x_2 - f_{34}x_1y_2 - f_{12}y_1x_2 + y_1y_2 + f_{14}x_2y_2 + \\ &\quad + f_{24}x_1y_1x_2y_2 \end{aligned}$$

and similarly for $\mathcal{E}_2$. It follows that

$$\begin{aligned} \chi(\mathrm{Four}(\mathcal{E}_1), \mathrm{Four}(\mathcal{E}_2)) &= \int ch((\mathrm{Four}(\mathcal{E}_1))^*) ch(\mathrm{Four}(\mathcal{E}_2)) \\ &= \mathrm{Pfaff}(F) + \mathrm{Pfaff}(G) + \epsilon^{ijkl} f_{ij} g_{kl} = \mathrm{Pfaff}(G - F)\,, \end{aligned}$$

where ϵ^{ijkl} is the Levi-Civita symbol. We conclude that the number of BRST states computed in section 2.3 is compatible with mirror symmetry.

4. Conclusion

Several issues remain to be understood. First of all, the relation between Heisenberg group representation and the noncommutativity of the A-model should be clarified. Also, it would be interesting to interpret these computations from the point of view of the analogue of the Künneth theorem for triangulated categories proved in [2]. Kontsevich has suggested that coisotropic branes should be considered as elements in the idempotent completion of the tensor square of the Fukaya category of E^τ. From this point of view, one only needs to understand the idempotents defining coisotropic summands, since the morphisms (and compositions!) of summands can be computed from these.

In summary, it has been anticipated that coisotropic branes appear as summands in the category generated by lagrangian branes and their shifts and sums, but a more direct treatment of these objects has been lacking. We hope that the description of morphisms in terms of representations of noncommutative algebras will help clarify the role of coisotropic branes in the Fukaya category and perhaps be relevant to the study of their deformations.[5] Eventually, one would like to know what an A-brane is!

[5]The deformation theory of classical coisotropic submanifolds has already proven quite interesting [7, 8].

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